Structural Reliability Assessment to Predict Probability of Failure of an Element in a System using Monte Carlo Simulation

Prakash R[1*], Ravindra R[2], Bhavana B[2]

[1]*Department of Mathematics, RV College of Engineering, Bengaluru, Karnataka, India*
[2]*Department of Civil Engineering, RV College of Engineering, Bengaluru, Karnataka, India*

Abstract

This paper presents Monte Carlo Simulation for reliability analysis of structural systems with respect to their failure and collapse. Structural reliability of a beam element used in building design was predicted using Monte Carlo Simulation. A beam was randomly selected in a structure, and its results in extended three-dimensional analysis of a building system were compared. The reliability index of the element is used to decide the failure of the element. Hence, this assessment can be used as a tool to proofread the design.

Keywords: *Monte Carlo simulation, Structural reliability, Probabilistic Reliability Assessment*

1.0 Introduction

Structures are designed to withstand loads due to wind, earthquakes and other natural calamities. Many a times, the effect of these loads is uncertain. Probability and statistical models are used to estimate the effect of the uncertainties. A structural design which satisfies safety and design requirements is considered safe. Structural reliability analysis is a higher level of proof reading the design. The objective of structural analysis is to quantitatively evaluate the structural safety based on probabilistic estimate [1]. Performance of an engineering system can be modelled mathematically along with conventional empirical formulae. Data pertaining to uncertainty are provided as input to the model through random variables for analysis. The uncertainties can be classified as; physical uncertainties which are associated with loading, properties and geometry of the materials; and statistical uncertainties which arise due to insufficient statistical information, e.g. a few compressive strength tests conducted on large concrete samples [2]. In case of insufficient sample size, it is difficult to assess the mean properties of the population. Hence, a mathematical model is used to estimate the properties with the available

[1]**Mail address: Prakash R, Associate Professor, Department of Mathematics, RV College of Engineering, Bengaluru – 59, Email: prakashr@rvce.edu.in, Ph: 9448024096*

results. In case of irreversible failures, determining the stage at which system failure may occur to a certain degree of accuracy is very difficult [3].

1.1 Structural reliability design

Structural reliability is the probabilistic estimation of the likelihood that a given system performs adequately for a specified period of time under the given operating conditions. On the other hand, the risk is known as the probability of failure under the same conditions [3]. The input and response of a system helps to analyse the risk of failure. In design of a structure or an element, load bearing capacity and the load imposed on the structure and its components decide its structural safety. The uncertainties in the design include the true strength of a structure and the predicted maximum load. These predictions are again based on the input data provided from the codal provisions. Hence, to ensure the safety of the system the probabilistic concepts help to determine the maximum load and failure of the elements in the system during the lifespan of the building.

In structural design an element like beam is analysed based on the stress induced and the strength. The stress and strength relationship are represented as shown in Fig. 1 [4].

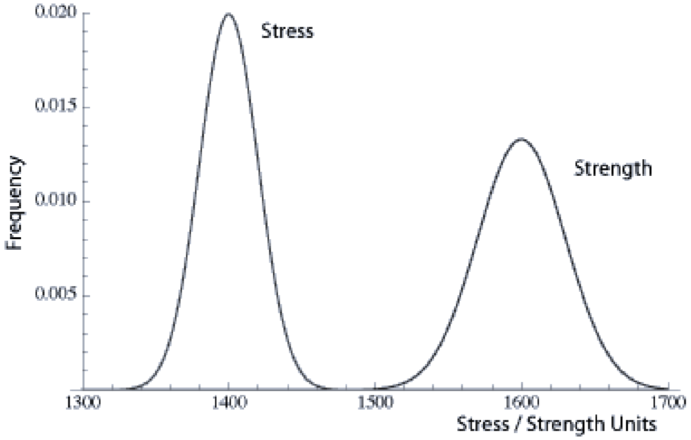

Fig. 1. Relationship between stress and strength - normally distributed [4]

It can be inferred from Fig 1 that the risk of failure of the individual element due to overstressing is minimal. In practice, the designers narrow down a list of failure parts of the structure after several design iterations. But in reality, these elements may fail due to overstressing and degradations. Hence, to avoid cumulative errors it is better to analyse the probability of failure of elements through ‘structural reliability assessment’.

Fig. 2. represents the interaction of stress and strength. The area under the interaction of both the curves gives the probability of failure which indicates that the probability of stress is higher than the strength. The motive of reliability analysis is to determine a multi-dimensional integral over an irregular region of failure [5].It is assumed that the probability of failure is a normally distributed curve.

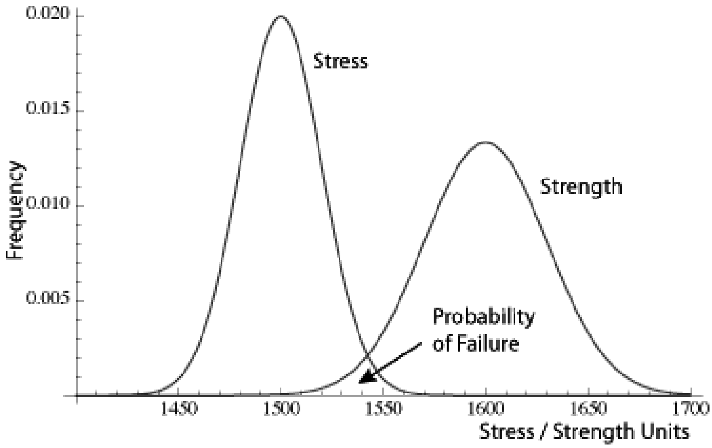

Fig. 2. Interaction of stress-strength [4]

Probability distribution function is used to determine the location and variation of the stress and strength. The margin of safety, Z=Y-X has mean and variance; X and Y are random variables of strength and stress:

a) Mean of the random variable, $\mu_z = \mu_y - \mu_x$
b) Standard deviation of the random variable, $\sigma_z^2 = \sigma_x^2 + \sigma_y^2$
c) Probability of failure $P(Y-X \leq 0) = P(z \leq 0) = \Phi(-\mu_z/\sigma_z)$
d) Reliability index, $\beta = \mu_z/\sigma_z$

$\Phi(Z)$ is the CDF of f(z) for z coordinate in a standard normal distribution. The failure region ($Z \leq 0$) of the random variable Z = R – Sis shown in Fig. 3.

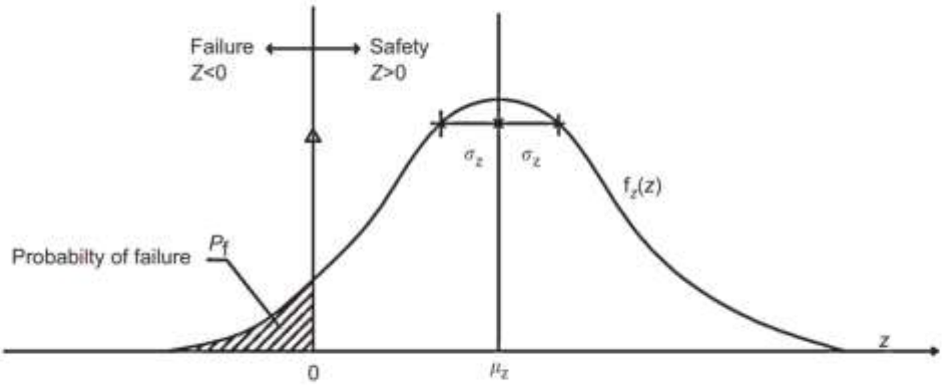

Fig. 3. Probability distribution of safety margin Z =R–S [3]

If incomplete information is available on the stress or strength then random variables can be generated from probability distribution as presented in case 1, 2 and 3.

Case 1. If the 'strength' is deterministic then probability distribution function (random variable X for stress) gives the probability of failure as expressed in equation (A). The failure occurs if strength is less than that of the induced stress (Fig. 4)

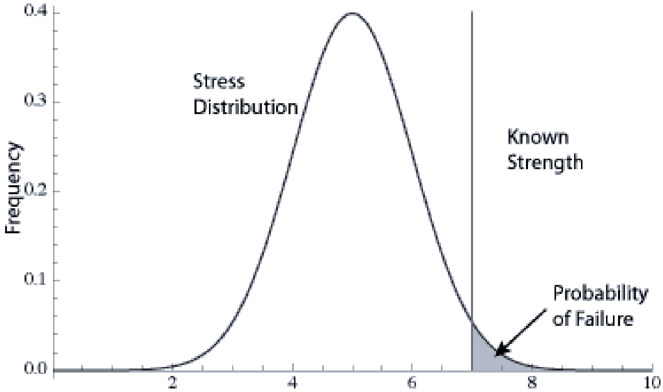

Fig. 4. Stress distribution for known values of strength [6]

$$P_f = \int_y^{\infty} f_X(x)dx \quad (A)$$

Case 2. If the 'stress' is deterministic then the probability distribution function (random variable X for strength) gives the probability of failure as given in equation (B). The failure occurs if stress exceeds the known strength as shown in Fig.5.

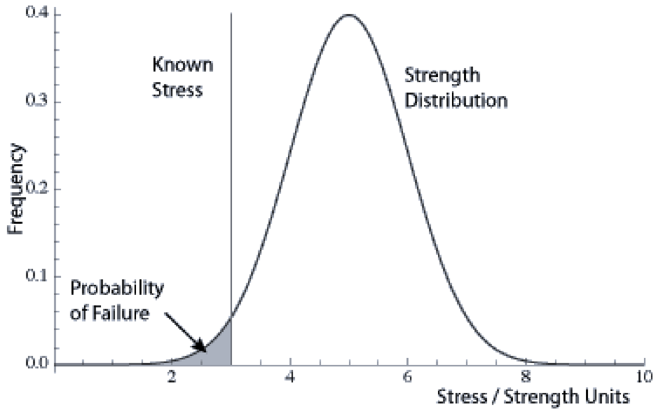

Fig. 5. Strength distribution for known values of stress [6]

$$P_f = \int_0^x f_Y\ (y)dy \qquad (B)$$

Case 3. If both 'strength and stress' are random variables, failure occurs when the stress is greater than the strength(Fig. 2). This is a special case where the stress and strength considerably fit a normal distribution. Hence, there is no need for double integration and the probability of failure is the difference between the two distributions. Safety margin is defined as the difference in the means of stress and strength as given in equation (1).

$$\text{Safety margin} = \mu_y - \mu_x \qquad (1)$$

1.2 Probabilistic reliability assessment

Probabilistic assessment is done for both the structure and individual elements. If an element fails then the entire structure becomes unsafe. This assumption can be treated as the probability of failure of the entire system occurs more frequently than in assessing the dependability of individual elements. The failure of that specific individual element has no influence on the adjoining member [5].It is achieved depending on the analysis of the reliability function (RF), equation (2).

$$RF(x) = R - S \qquad (2)$$

Where x is the random vector of input variables like load effects, geometry of the structure, mechanical properties or environmental factors on the structure, R is the 'structural resistance' or capacity and **S**is the'effect of the load' or demand.

Reliability condition is given by in equation (3).

$$S \leq R \rightarrow R - S \geq 0 \rightarrow RF(x) \geq 0 \qquad (3)$$

when S > R, the condition (2) fails and it is an unfavourable state. The probability of failure P_f is obtained using equation (2) and it is expressed in equation (4).

$$P_f = P(R - S < 0) = P(RF(X) < 0) = P(Z \leq 0) \quad (4)$$

Also it is observed that P_d,probability of design failure holds the following inequality:

$$P_f \leq P_d \quad (5)$$

1.3 Conventional measures of reliability

The measures of reliability can also be defined in terms of demand Y and capacity X.

1.3.1Factors of safety (FOS)

The safety and risk of a structure is analysed based on the allowable 'factor of safety' (FOS) denoted by Z and is defined as the ratio of the assumed nominal values of demand Y and capacity X.

$$Z = X / Y \quad (6)$$

1.3.2 Safety margin (S)

The difference between the capacity and demand of the system is known as safety margin and is given by the following relation.

$$S = X - Y \quad (7)$$

The reliability index is defined as:

$$\beta = \mu / \sigma \quad (8)$$

where σ and μ are the standard deviation and mean of the reliability function RF(x). Probabilistic reliability assessment can also be performed at the level of reliability indexβ and is equivalent to the probability of failure $\mathbf{P_f}$ and is given by equation (5)[7].

$$\beta = -\Phi^{-1}(P_f) \quad (9)$$

The reliability index is categorised as follows in Table 1.

Table 1. Assessment of reliability state and maintenance action based on probability index [8]

Reliability State	**5**	**4**	**3**	**2**	**1**
Reliability index (β)	β >9	9 > β > 8	8 > β > 6	6> β > 4.6	B <4.6
Reliability Attribute	Excellent	Very good	Good	Fair	Unacceptable
Maintenance action	Nil	Precautionary inspection	Detail inspection	Possible strengthening	Rehabilitation required

2.0 Monte Carlo Simulation

Simulation is explored for solving reliability problems for which analytical methods are difficult to apply [9]. Monte Carlo simulation (MCS) is a numerical method which is used to study the distribution of a function of multiple random variables, to simulate the performance or behaviour of a system, and to compute the probability of failure or reliability of a system or a component.MCS has wide range of applications involving probability and it is used to find the solution of a probabilistic problem of complex nature [10].MCS differs from probabilistic distributions in that it does not proceed analytically, instead it relies on repeated random sampling to produce numerical results. However, MCS has a disadvantage that it requires a large number of samples to handle small probabilities, resulting in high computational cost [7].

Soong and Grigoriu [11] presented that the relationship between $P_{\text{approx.}}$(MCS) and P_{actual} (failure probability) as shown in equation (10):

$$E\,(P_{\,approx.}) = P_{actual} \quad (10)$$

$$\sigma^2_{P_{approx}} = (1/N)\,(P_{\text{actual}} - (1-P_{\text{actual}})) \quad (11)$$

$$V_{p\ \text{approx.}} = \sqrt{(1 - P_{\text{actual}})\,/(N \times P)} \quad (12)$$

$$N = (1-P_{\text{actual}})\,/\,v^2_{p\ \text{approx}}\ x\ P_{\text{actual}} \quad (13)$$

where N is the total number of samples,$\sigma^2_{p\ \text{approx}}$, E($P_{approx}$) and $V_{p\ approx.}$, are respectively the variance, expected value and the coefficient of variation of estimated probability. Increase in N reduces the variance and dispersion of estimation as per MCS, making the results less uncertain; however, a large number of samples are required, making the method difficult to apply. In general, traditional Monte Carlo sampling is a poor approximation of the tail, and a small error in the tail leads to a large error in the estimated failure probability. The type of algorithm used to generate more random numbers near the tail changes the spread of the random numbers, generating more random numbers at certain angles or regions such as the tail region.

In the present study a simplified MCS was used for estimate the smallest probability failure that can be efficiently programmed and implemented to solve complex problems. The random variable can follow any distribution. For example if the performance function, LSF=R-S, with R, strength resistance of log-normal distribution function with standard deviation σ_R, mean μ_R and **S**(load effect) with mean μ_S and standard deviation σ_S. MCS can be carried out efficiently as follows:

Let P_i be the random number, Z be the Normal variate, with a set of standard-normal random variables Z_i generated from $P_i$using the inverse cumulative distribution function (CDF), ϕ and it can be expressed in the following equation.

$$Z= \phi^{-1} (P_i) \tag{14}$$

The corresponding log-normal variables, X_i are generated using the relationship between standard-normal distribution and log-normal distribution as given in equation (6):

$$\left.\begin{aligned} V_R &= \sigma_R / \mu_R \\ \sigma^2_{\ln(R)} &= \ln (V^2_R+1) \\ \mu_{\ln(R/S)} &= \ln (\mu_R) - \tfrac{1}{2} \times \sigma^2_{\ln(R)} \\ r_i &= \exp. (\mu_{\ln (R)} + Z_i \, \sigma^2_{\ln(R)}) \end{aligned}\right\} \tag{15}$$

Similarly, the random numbers are generated for S and the standard deviation, mean and variance are calculated as expressed in equation (15).

2.1 Reliability assessment using MCS – a case example

A reinforced concrete framed four storey structure was modelled and analysed in ETABS (Extended 3D Analysis of Building System)software (Fig 6), as per the codal provisions IS 456:2000[12] and IS 1893 (part 1):2002 [13] and the required design parameters were calculated and provided as input data.

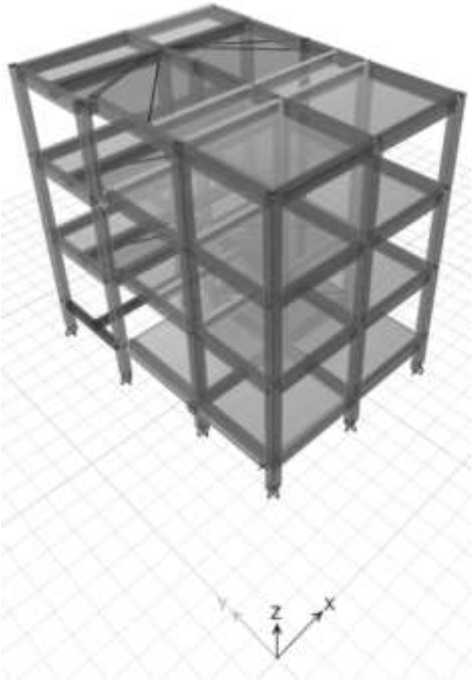

Fig. 6. Isometric view of the building in ETABS

Structural members like beams, columns and slabs of dimensions (Table 2) were designed, analysed and evaluated for minimum design requirements and safety as per IS 456:2000.

Table 2. Details of the structural components

Colour Code for representing structural member	Structural Components	Dimensions of structural Components
	Beam	230 x 450,230 x 300 mm
	Column	230 x 450, 230 x 230 mm
	Slab	125 mm

Margin of safety for a flexural member is given by: M=R-S, where R is the flexural strength and S is the load on the member. In the present study one beam element was considered for reliability assessment. The steps involved in MCS include generating the random number (sample), applying the numerical model, the number of iterations (simulations) - N required and data analysis. The steps in reliability analysis using MCS is presented in Fig. 7.

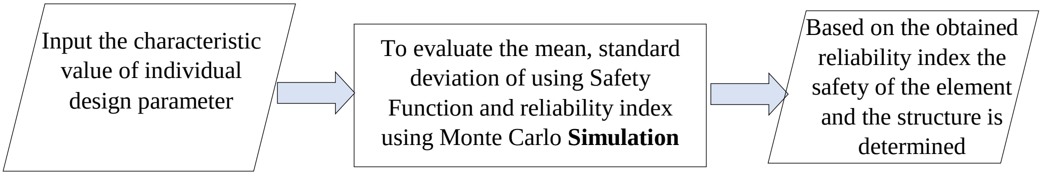

Fig. 7. Steps in MCS - Reliability analysis to determine the reliability index

The super dead load and live load calculations for design are considered as per Table 3 and other design parameters are mentioned Table 4.

Table 3. Loads parameter for Design

Sl. No	Parameter	
1.	**Unit weight (**kN/m^2**) of building materials as per IS 875 – PART 1 (1987)**	
	i. Ceiling plaster	0.35
	ii. Floor finish	1.00
	iii. Water proofing at terrace	2.04
	iv. Floor finish at terrace (Mangalore tiles)	0.02
2.	**Density (**kN/m^3**) of building materials as per IS 875 – PART 1 (1987)**	
	i. Masonry wall	20
	ii. Reinforced cement concrete	25
3. i. i.	**Imposed load (**kN/m^2**) as per is 875 – part 2 (1987)**	
	i. Live load at typical floor levels	2.00
	ii. Live load on terrace	1.50

Table 4. Other design parameters

Other design parameters	
1.	Wind load : not designed for wind load
2.	Location : Bangalore
3.	Earthquake load: As per IS 1893 (part 1) :2002
4.	Soil type : Type II, medium as per IS 1893 (part 1) :2002
5.	Allowable bearing pressure: 250 kN/m^2
6.	Walls : 230 mm thick brick masonry walls
7.	Beams :230 x 450 mm (primary beam), 200 x 300 mm (secondary beams)
9.	Columns: 230 x 450 mm, 230 x 230 mm
10.	Slab thickness: 125 mm

The frame was analysed by equivalent static lateral force method. The input data in ETABS for the analysis of the frame by equivalent static lateral force method is followed as per IS 1893(Part 1):2002.

2.2 Calculation of reliability index

A beam is typically having dead load W_d and live load W_l. The loads as calculated as per IS 456:2000. A st, d, f_y, b, f_{ck} are assumed deterministic. The variable sf_y, W_l, f_{ck} are considered to be normally distributed. A beam of span L= 3.08m, width b=230 mm and depth = 600 mm is simply supports the ends. A st, d, f_y, b, and f_{ck} are deterministic variables, where,

A_{st} = Area of steel in sq.mm

d = depth of the beam in mm = 600 mm

f_y = Yield stress in steel in N/mm^2= 500 N/mm^2

b = width of the beam = 230 mm

f_{ck} = Grade of concrete in N/mm^2 = 30 N/mm^2 (M30 Grade)

W_l =Wall load on the beam in kN

The load and flexural resistance on the beam is as follows

Resistance R=f_y *Ast *d [1-(0.77* f_y *Ast)/(b*d* f_{ck})](16)

Load S= [(W_l+ W_d)*L*L] / 8

The characteristic values A st and W_l are obtained from the analysis of the model in ETABS. A st =350 mm^2 for one of the beams of length L=3.08 m with width b =230 mm, depthd=600 m and W_l=11.4 kN. On substituting the above parameters in the equations(16) and (17),we get,

R= f_y *21*104 [1-(0.001953* f_y/ f_{ck})]..... kN-m , S= 6.136+1.186 W_l ..kN-m

$\mu_{f y}$ = $f_{y/}$ [1-1.645 * Covariance of f_y], $\mu_{f ck}$ = f_{ck} /[1-1.645 * Covariance of f_{ck}]

μ_{Wl} = W_l/[1-1.645 * Covariance of W_l]

where $\mu_{f y}$, $\mu_{f ck}$,μ_{Wl} are mean of f_y, f_{ck}, W_l respectively . $\sigma_{f y}$, $\sigma_{f ck}$, σ_{Wl} are standard deviation of f_y, f_{ck}, W_l respectively.

Generally, the covariance value is 10% for design of structure using reliability. Here the strength is under estimated and loads are over estimated.

a. Mean value of the parameters:

$\mu_{f y}$ = 500/[1-1.645 * 0.1] = 595.24 N/mm^2

$\mu_{f ck}$ = 30/[1-1. 645 * 0.1] = 35.71 N/mm^2

μ_{Wl} = 11.40/[1-1. 645 * 0.1] = 13.57 kN

b. Standard deviation of the parameters:

$\sigma_{f\,y}$ = Covariance of $f_{y} * \mu_{f\,y}$ = 0.1 * 595.24 = 59.25 N/mm^2

$\sigma_{f\,ck}$ = Covariance of $f_{ck} * \mu_{f\,ck}$ = 0.1 * 35.71 = 3.57 N/mm^2

σ_{Wl} = Covariance of $W_l * \mu_{Wl}$ = 0.1 * 13.57 = 1.36 kN

Let P_i = Random numbers and Z be the Normal variate as given in equation

$Z=(X-\mu)/\sigma$ or $Z=\phi^{-1}(P_i)$ (16)

Random variable using random number (X) as given in equation

$X=\mu+Z\sigma$

where, μ= mean value, σ= standard deviation of the design parameter.X and Z are calculated as shown in Table 5.

Table 5. Computation of random variable (Z) from random numbers (X)

Statistical parameters		**f_y = Yield stress in steel in N/mm^2**	**f_{ck} = Grade of concrete in N/mm^2**	**W_l = Wall load on the beam in k N**
Mean value	μ	595.24	35.71	13.57
Standard deviation	σ	59.25	3.57	1.36
Random numbers	Normal variate (Z)	Random variable using random number (X)		
P_i	$Z=\phi^{-1}(P_i)$	$X_{f\,y}=\mu+Z\sigma$	$X_{f\,k}=\mu+Z\sigma$	$X_{Wl}=\mu+Z\sigma$
0.39	-0.28	578.65	34.71	13.19
0.66	0.41	619.53	37.17	14.13
0.57	0.17	605.31	36.32	13.80
0.73	0.61	631.38	37.89	14.40
0.12	-1.17	525.92	31.53	11.98
0.24	-0.71	553.17	33.18	12.60
0.18	-0.91	541.32	32.46	12.33
0.75	0.67	634.94	38.10	14.48
0.81	0.88	647.38	38.85	14.77
0.35	-0.38	572.73	34.35	13.05

Table 6 shows the computation of resistance, load capacity, margin of safety and mean and standard deviation of the characteristic values.

Table 6. Computation of Resistance (R), Load capacity (S), Margin of Safety (M), Mean and standard deviation value of M and S

Resistance (R)	**Load capacity (S)**	**Margin of Safety (M)**	
R= X_{fy} *21*10^4 [1-(0.001953* X_{fy}/ X_{fck})]	S= 6.136+1.186 X_{Wl}	M= R-S	
117.56	21.78	95.78	Mean value μ
125.87	22.89	102.98	of Margin of Safety
122.98	22.50	100.47	97.96
128.28	23.21	105.06	
106.85	20.34	86.50	
112.38	21.08	91.30	Standard deviation σ
109.98	20.76	89.21	of Margin of Safety
129.00	23.31	105.69	6.92
131.53	23.65	107.88	
116.36	21.62	94.74	

c. Reliability Index (β)

$$\beta = \mu_{m} / \sigma_{m} = 14.46$$

3.0 Results and Discussion

From Table 1we can assess the grade of reliability state and maintenance action on the structural member as shown in Table 7

Table 7. Computation of characteristic value from random numbers

Reliability state	5
Reliability index, β	β >9
Attribute for reliability	excellent
Maintenance action	No action

The reliability value β is more than 9; from this we can conclude that the beam is safe in design (from Table 1).The same has been verified using ETABS software using the MCS.

4.0 Conclusion

i. The aim of any structural design is to make sure that the safety and economy of the structure operating in a given environment is achieved (Capacity (C)>Demand (D)).

ii. The results propose that probabilities of serviceability failure are consistent across a certain range of beam spans. The span-to-depth ratio serviceability requirements as that of the ETABS results.

iii. Using a smaller volume of samples the simplified method of MCS estimates the probability of failures to an equivalent level of accuracy.

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References

1. M Liu, Reliability analysis of seismic shear-type structures, *Advances in Intelligent and Soft Computing*,114, 181–187,2012
2. M H Faber, Basics of structural reliability. Swiss Federal Institute of Technology ETH2009.
3. R Camposinhos, Dimension stone design – partial safety factors: a reliability-based approach, *Proceedings of Institution of Civil 2012 Engineers: Construction Materials*,165(3), 145-159, 2012
4. F Schenkelberg, Establishing product reliability goals, *Proceedings Annual Reliability and Maintainability Symposium (RAMS-IEEE)*, 1-6, 2013,
5. G Li K Zhang, A combined reliability analysis approach with dimension reduction method and maximum entropy method, *Structural and Multidisciplinary Optimization*, 43(1),121–134, 2011
6. V Jakub, M Krejsa, Probabilistic Reliability Assessment of Truss Construction in Matlab Software Platform, *Civil Engineering Series*,14(1),1-10, 2014
7. M S Far, Y Wang, Approximation of the Monte Carlo Sampling Method for Reliability Analysis of Structures, *Mathematical Problems in Engineering*, 16, 1-9, 2016

8. T Guo, A Q Li, C Q Miao, Combined numerical simulation method and its application in probability assessment of Runyang bridg, *Structural health monitoring and intelligent Infrastructure,* Second edition, Taylor and Francis Group, London, 2006

9. F Biondini, Use of Simulation in Structural Reliability, *Structures Congress 2008, American society of Civil Engineers*, ISBN (print): 9780784410165, 2008.

10. A H S Ang, W H Tang, Probability Concepts in Engineering, Second Edition, *John Wiley & Sons*, 2007

11. T T Soong, M Grigoriu, Random Vibration of Mechanical and Structural Systems, *Prentice Hall, New York, New York, USA*, 1993.

12. IS 456:2000. Plain and reinforced concrete-code of practice, Fourth Revision, Bureau of Indian standards, New Delhi.

13. IS 1893 (Part 1):2016. Criteria for earthquake resistant design structures, Sixth revision, Bureau of Indian standards, New Delhi.